

# PERTURBING PLA

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**ABSTRACT.** We proved earlier that every measurable function on the circle, after a uniformly small perturbation, can be written as a power series (i.e. a series of exponentials with positive frequencies), which converges almost everywhere. Here we show that this result is basically sharp: the perturbation cannot be made smooth or even Hölder. We discuss also a similar problem for perturbations with lacunary spectrum.

## 1. INTRODUCTION

**1.1. Functions representable by analytic sums.** Let a power series converge almost everywhere on the circle  $\mathbb{T}$  to a function  $g$ :

$$g(t) = \sum_{n \geq 0} c(n) e^{int} \quad (1)$$

It follows from the Privalov uniqueness theorem, that any  $g$  may have at most one such decomposition. An analogy with the classical Riemannian theory suggests that  $c(n)$  are the Fourier coefficients, whenever  $g$  is integrable.

Quite surprisingly, this is not the case: a few years ago we constructed an  $L^2$ -function  $g$  on  $\mathbb{T}$  which admits the representation (1) but

$$\sum |c(n)|^2 = \infty.$$

Later we proved that such a function even can be smooth.

The space of functions  $g$  which admit an “analytic” representation (1) we named PLA. The classic PLA-part of  $L^2(\mathbb{T})$  is the set of functions whose Fourier series contains exponentials with non-negative frequencies only, namely the Hardy space  $H^2$ . This set is “small”, in particular it is nowhere dense. In contrast the “non-classic” part is dense. Moreover, the following equality is true:

$$L^0 = \text{PLA} + C(\mathbb{T}) \quad (2)$$

which means that every measurable finite function  $f$  can be decomposed as a sum

$$f = g + h \quad (3)$$

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where  $g \in \text{PLA}$  and  $h$  is continuous. Further, one can replace  $C(\mathbb{T})$  with  $U(\mathbb{T})$ , the space of uniformly convergent Fourier series, and one can require from  $h$  to have arbitrarily small norm (the norm in  $U(\mathbb{T})$  being the supremum of the modulus of the partial sums of the Fourier expansion). The described results are proved in [KO.06, KO.07].

Our first result is that the equality (2) is close to best possible: one can not replace the second summand by a space of functions which possess any smoothness, like Hölder or Sobolev one. We state the result in the following form. Given a sequence  $\omega = \{\omega(n)\}$ ,  $0 < \omega(n) \nearrow \infty$ , denote:

$$\mathcal{H}_\omega = \left\{ h : \sum |\widehat{h}(n)|^2 \omega^2(n) < \infty \right\}. \quad (4)$$

**Theorem 1.** *For any  $\omega$  the sum  $\text{PLA} + \mathcal{H}_\omega$  does not cover neither  $L^0(\mathbb{T})$ , nor even the Wiener algebra  $A(\mathbb{T}) = \widehat{l_1(\mathbb{Z})}$ .*

This theorem will be proved in §3.

**1.2. Menshov spectra revisited.** The classic Menshov representation theorem (1940), see [B64, §XV.2] says that every function  $f \in L^0(\mathbb{T})$  admits representation by a trigonometric series which converges a.e.:

$$f = \sum_{n=-\infty}^{\infty} c(n) e^{int}. \quad (5)$$

This representation is non-unique, as follows from another remarkable result of Menshov's proved much earlier (1916): there is a non-trivial trigonometric series which converges to zero almost everywhere. Menshov's construction for the representation reveals the non-uniqueness phenomenon in a stronger form: one can avoid in (5) using any finite and even some infinite sets of harmonics. This leads to the following definition, see [KO.01]:

**Definition.** A sequence  $\Lambda \subset \mathbb{Z}$  is called a Menshov spectrum if every function  $f \in L^0(\mathbb{T})$  can be decomposed to a series (5) in which only frequencies from  $\Lambda$  may appear with non-zero amplitudes.

In this terminology, Menshov's theorem states that  $\mathbb{Z}$  is a Menshov spectrum. There are many results that show that Menshov spectra could be quite sparse. For example Arutyunyan [A85] showed that any symmetric set which contains arbitrarily long intervals is a Menshov spectrum. In other words, the set

$$\bigcup_{n=1}^{\infty} [a_n, a_n + n] \cup [-a_n - n, -a_n]$$

is a Menshov spectrum, no matter how fast do the  $a_n$  grow. Of course, such sets can be extremely sparse. Here we wish to compare to the following sparseness result, taken from [KO.01]:

Given a sequence

$$\omega(k) = o(1) \tag{6}$$

one can construct a sequence  $\lambda(k) \in \mathbb{Z}^+$  with  $\lambda(k+1)/\lambda(k) > 1 + \omega(k)$ , such that  $\Lambda = \{\pm \lambda(k)\}$  is a Menshov spectrum.

The condition (6) is sharp: a Menshov spectrum cannot be lacunary in Hadamard sense. Further, the symmetry condition is also essential. Indeed, Privalov's uniqueness theorem implies that the set  $\mathbb{Z}^+$  is not a Menshov spectrum. See [KO.01] for details on all these claims.

One may now ask: how many negative frequencies one should add to  $\mathbb{Z}^+$  in order to get a Menshov spectrum? According to the theorem above an extra set with gaps of any sub-exponential growth could be sufficient. Our second result is that this result is close to the best possible one: super-exponential growth is not sufficient.

**Theorem 2.** Let  $Q := \{q(k)\} \subset \mathbb{Z}^+$  satisfy the condition

$$\frac{q(k+1)}{q(k)} \rightarrow \infty. \tag{7}$$

Then the set  $\Lambda = \mathbb{Z}^+ \cup \{-Q\}$  is not a Menshov spectrum.

Theorem 2 can be reformulated in the language of theorem 1. Let

$$\mathcal{L}_Q = \{f \in L^1 : \hat{f}(n) = 0 \ \forall n \notin Q\}.$$

Then

**Theorem 2'.** With the same  $Q$  as in theorem 2,  $\text{PLA} + \mathcal{L}_{-Q} \neq L^0$ .

The equivalence of theorems 2 and 2' follows by taking the Menshov representation of  $f$  and making the positive part into a PLA function and the negative part into an  $\mathcal{L}_{-Q}$  function. This requires Plessner's theorem and some standard facts on lacunary trigonometric series — we fill these details in §4.

The formulation of theorem 2' leads to a natural generalisation. Can one find a function  $f \notin \text{PLA} + \mathcal{L}_{-Q}$  for all superexponential  $Q$  simultaneously? We present a weakened version of this

**Theorem 3.** For a function  $\ell(n) \rightarrow \infty$  there is a function  $f$  such that  $f \notin \text{PLA} + \mathcal{L}_{-Q}$  for any  $Q$  satisfying

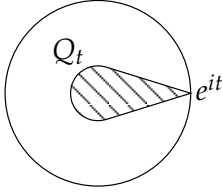
$$\frac{q(k+1)}{q(k)} > \ell(q(k)).$$

We remark that, as in theorem 1, the  $f$  of theorem 3 may be taken to be in the Wiener algebra  $A(\mathbb{T})$ .

Theorem 3 is clearly stronger than theorem 2', and hence also from theorem 2. On the other hand, the proof is also more technical. Hence we first prove theorem 2 in §4, and only afterwards give the proof of theorem 3 in §5.

## 2. LEMMAS

In this section we introduce some notation and lemmas which will be used for the proof of all three theorems. For a PLA-function  $g$  the coefficients  $c_n$  in the expansion (1) are unique, so we will denote them by  $\widehat{g}^{\text{PLA}}(n)$ . Below we denote by  $g$  any PLA-function with  $\widehat{g}^{\text{PLA}}(0) = 0$ . We will use the following notations



$$\begin{aligned} g^*(t) &:= \sup_N \left| \sum_{n < N} \widehat{g}^{\text{PLA}}(n) e^{int} \right| \\ G(z) &:= \sum \widehat{g}^{\text{PLA}}(n) z^n, |z| < 1 \\ Q_t &:= \text{conv}(\{e^{it}\} \cup \{z : |z| < \tfrac{1}{2}\}). \end{aligned} \tag{8}$$

This  $Q_t$  is often called the Privalov ice-cream cone at  $e^{it}$ . We always denote by  $E$  a measurable subset of  $\mathbb{T}$ ; by  $|E|$  its Lebesgue measure. By  $\|\cdot\|_2$  we denote the norm in  $L^2(\mathbb{T})$ .

**Lemma 1.** *If  $g^*(t) = A$  then  $|G(z)| \leq 3A$  for all  $z \in Q_t$ .*

*Proof.* Without loss of generality one may assume  $t = 0$ . We now apply Abel's summation formula to get, for any  $|z| < 1$ ,

$$G(z) = \sum_{n=0}^{\infty} \widehat{g}^{\text{PLA}}(n) z^n = \sum_{n=0}^{\infty} (z^n - z^{n+1}) \sum_{k=0}^n \widehat{g}^{\text{PLA}}(k)$$

so

$$|G(z)| \leq A \sum_{n=0}^{\infty} |z^n - z^{n+1}| = \frac{A|1-z|}{1-|z|}$$

but in  $Q_0$  one has  $|1-z|/(1-|z|) \leq 3$ , with the maximum achieved at  $z = -\frac{1}{2}$  (the exact value of the constant 3 will play no role in what follows).  $\square$

**Lemma 2.** *There is some universal constant  $c_1 > 0$  such that for every  $K > 0$  there is a number  $\varepsilon = \varepsilon(K)$  such that if*

$$\|1 + g\|_2 < \varepsilon$$

*then*

$$|\{t : g^*(t) > K\}| > c_1.$$

We remark that in fact  $c_1$  may be taken to be  $1/2$  or any number smaller than 1, but this requires an extra argument that we prefer to skip. The dependency between  $\varepsilon$  and  $K$  will turn out to be  $\varepsilon \approx 1/K^2$ . The power can be reduced arbitrarily close to zero (e.g.  $\varepsilon \approx K^{-0.0001}$ ) at the price of making  $c_1$  smaller (we will not need all these in this paper).

*Proof.* The proof is a simple variation on the proof of Privalov's uniqueness theorem [K80, §D.III]. Let  $A > 1$  be some sufficiently large parameter to be fixed later, and let  $E \subset [0, 2\pi]$  be the set of  $t$  satisfying the following two requirements

$$\begin{aligned} |1 + g(t)| &< A\varepsilon \quad \forall t \in E \\ |g^*(t)| &\leq K \quad \forall t \in E. \end{aligned} \tag{9}$$

Assume by contradiction that  $|\{t : g^*(t) > K\}| \leq c_1$ . Then we may assume that

$$|E| > 1 - \frac{1}{A^2} - c_1$$

since Markov's inequality gives us

$$|\{t : |1 + g(t)| \geq A\varepsilon\}| < \frac{1}{A^2}.$$

(the fact that the power is 2 will play no role in the argument).

Next, recall that  $G$  is the "extension" of  $g$  into the disk  $\{|z| \leq 1\}$  defined by (8) whenever the sum converges, which is on all of  $\{|z| < 1\}$  and almost everywhere on  $\{|z| = 1\}$ , since  $g$  is in PLA. By Abel's theorem, if  $\sum \widehat{g}^{\text{PLA}}(n)e^{int}$  converges then  $G(z) \rightarrow G(e^{it})$  when  $z$  converges to  $e^{it}$  non-tangentially [Z68, §3.14]. Assume therefore, without loss of generality, that the convergence  $G(z) \rightarrow G(e^{it})$  is uniform on  $E$  and that  $E$  is closed (if it is not, use Egoroff's theorem to find an  $E' \subset E$  satisfying the requirement and having large measure,  $|E'| > 1 - A^{-2} - c_1$ ).

Examine the Privalov domain over  $E$ , namely

$$P = \bigcup_{t \in E} Q_t.$$

See figure 1 which demonstrates a Privalov domain for a Cantor set. By the above,  $G$  is continuous on  $P$ . Next examine the function  $\ell := \log |1 + G|$ . It is subharmonic on  $P$ , and continuous on  $\overline{P}$  (in the sense that allows the value  $-\infty$ ). Therefore

$$\ell(0) \leq \int_{\partial P} \ell(z) d\Omega(z) \tag{10}$$

where  $\Omega$  is the harmonic measure of  $P$  from 0 (which is clearly a point of  $P$ ). For background on the harmonic measure (and especially its construction using Brownian motion) see the book [B95].

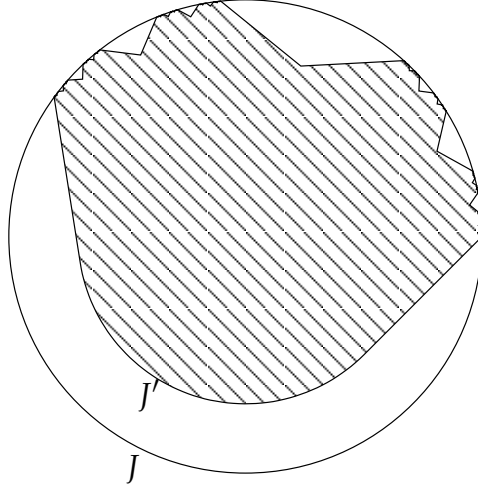


FIGURE 1. A Privalov domain for a Cantor set.

To use (10) we first note that  $G(0) = 0$  so  $\ell(0) = 0$ . Now examine the boundary of  $P$ . We write  $\partial P = E \cup I$ . On  $E$  we have  $\ell < \log A\varepsilon$ . On  $I$  we apply lemma 1 to see that  $\ell \leq \log 3K$ . Finally we need to estimate the harmonic measure  $\Omega$  of  $I$  in the domain  $P$ . Every interval  $J$  in the complement of  $E$  corresponds to a piece  $J'$  of  $I$  — usually to just two straight lines from the edges of  $e^{iJ}$ , but sometimes also to a piece of  $\{|z| = \frac{1}{2}\}$ . One  $J$  and  $J'$  of the second kind (i.e. with a piece of  $\{|z| = \frac{1}{2}\}$ ) are noted in figure 1. Either way, a straightforward calculation shows that the probability that Brownian motion starting from 0 hits  $J'$  before leaving the disk is  $\leq C|J|$  and hence

$$\Omega(I) = \sum \Omega(J') \leq C \sum |J| = C|E^c| < C(A^{-2} + c_1).$$

Define therefore

$$A = 2C^{-1/2} \quad c_1 = \frac{1}{4C}$$

and get that  $\Omega(I) < \frac{1}{2}$  and hence that  $\Omega(E) > \frac{1}{2}$ . With the estimates above for  $\ell$  and (10) we get

$$0 = \ell(0) \leq \Omega(E) \log A\varepsilon + \Omega(I) \log 3K < \frac{1}{2} \log A\varepsilon + \log 3K$$

which leads to a contradiction if only  $\varepsilon$  is sufficiently small.  $\square$

*Remark.* It might be worthwhile to compare this lemma to lemma 2.6 in [N93], which is also proved by Privalov's approach.

## 3. PROOF OF THEOREM 1.

Let  $\{b(k)\}$  ( $k = 1, 2, \dots$ ) be a fast decreasing sequence of positive numbers, such that

$$\sum_{j>k} b(j) = o\left(b(k)\varepsilon\left(\frac{k}{b(k)}\right)\right) \quad \forall k \quad (11)$$

where  $\varepsilon(K)$  are from lemma 2.

Recall the sequence  $\omega$  going to infinity from the statement of the theorem. Given  $\omega$ , choose a fast increasing sequence  $\{n(k)\}$  of integers so that

$$b(k)\varepsilon\left(\frac{k}{b(k)}\right)\omega(n(k)) \rightarrow \infty \quad (12)$$

Set

$$f(t) := \sum_{k>0} b(k)e^{-in(k)t}.$$

We claim that the function  $f$  does not belong to  $\text{PLA} + \mathcal{H}_\omega$ . Take therefore any  $h \in \mathcal{H}_\omega$  and let  $q := f - h$ . We need to show that  $q \notin \text{PLA}$ . Fix a (large) number  $N$ . Denote

$$\begin{aligned} f'(N; t) &= \sum_{k<N} b(k)e^{-in(k)t} \\ f''(N; t) &= \sum_{k>N} b(k)e^{-in(k)t} \end{aligned}$$

so

$$f = f' + b(N)e^{-in(N)t} + f''.$$

Similarly, let

$$h = h' + h''$$

where

$$h' := \sum_{|n|<n(N)} \widehat{h}(n)e^{int}.$$

Clearly

$$|f''(t)| \leq \sum_{j>k} b(j)$$

and

$$\|h''\|_2 < \|h\|_{\mathcal{H}_\omega} / \omega(n(N)). \quad (13)$$

Examine now the function  $g := [1/b(N)](f' - h' - q)e^{in(N)t}$ . It is in PLA since  $q \in \text{PLA}$  (we argue by contradiction here) and  $e^{in(N)t}(f' - h')$  is an analytic polynomial. In other words

$$\widehat{g}^{\text{PLA}}(k) = \frac{1}{b(N)} \left( \widehat{f}'(k - n(N)) - \widehat{h}'(k - n(N)) - \widehat{q}^{\text{PLA}}(k - n(N)) \right)$$

and  $\widehat{g}^{\text{PLA}}(0) = 0$ . Hence we may apply lemma 2. For the  $L^2$  norm we can write

$$\begin{aligned} \|1 + g\|_2 &= \left\| \frac{e^{in(N)t}}{b(N)} \left( b(N)e^{-in(N)t} + f' - h' - q \right) \right\|_2 = \\ &= \frac{1}{b(N)} \| -f'' + h'' \|_2 \leq \\ \text{By (11) and (13)} \quad &\leq \frac{1}{b(N)} \left( o \left( b(N)\varepsilon \left( \frac{N}{b(N)} \right) \right) + \frac{\|h\|_{\mathcal{H}_\omega}}{\omega(n(N))} \right) \\ \text{By (12)} \quad &= o \left( \varepsilon \left( \frac{N}{b(N)} \right) \right) \end{aligned}$$

(where the  $o$  is allowed to depend on  $\|h\|_{\mathcal{H}_\omega}$ ). So for  $N$  sufficiently large the  $o$  is smaller than 1, and the lemma gives that

$$\left| \left\{ t : g^*(t) > \frac{N}{b(N)} \right\} \right| > c_1. \quad (14)$$

At this point we only need to go back from  $g$  to  $q$ , so we need to estimate the contributions of  $f'$  and  $h'$ .  $f'$  is straightforward as

$$\sup_k \left| \sum_{j < k} \widehat{f' e^{in(N)t}}(j) e^{ijt} \right| \leq \sum_k |\widehat{f}(k)| < C. \quad (15)$$

For  $h'$  we use Carleson's theorem [C66, L04] for both  $h^+$  and  $h^-$  defined by

$$h^+ = \sum_{n \geq 0} \widehat{h}(n) e^{int} \quad h^- = \sum_{n < 0} \widehat{h}(n) e^{int}$$

and get that both expansions converge almost everywhere. This gives a set  $E$  with  $|E^c| \leq \frac{1}{2}c_1$  such that

$$\left| \sum_{n=0}^k \widehat{h}(n) e^{int} \right| \leq C \quad \left| \sum_{n=-k}^{-1} \widehat{h}(n) e^{int} \right| \leq C \quad \forall t \in E, \forall k. \quad (16)$$



For  $h'e^{in(N)t}$ , the analogous sum is bounded by either a sum of two terms from (16), or by a difference of two, and in both cases we get

$$\left| \sum_{j < k} \widehat{h'e^{in(N)t}}(j) e^{ijt} \right| \leq 2C. \quad (17)$$

This proves the theorem: since  $g^*$  is large (14) and  $f'$  and  $h'$  are bounded (15), (17), we get

$$q^* \geq N - C$$

on a set of measure  $> \frac{1}{2}c_1$ . Since  $N$  was arbitrary and  $C$  depends only on  $h$ , this proves that  $q \notin \text{PLA}$ .  $\square$

*Question.* Does  $\text{PLA} + A(\mathbb{T})$  cover  $C(\mathbb{T})$ ?

#### 4. PROOF OF THEOREM 2

Let us start by showing that theorem 2 is equivalent to theorem 2' (this will also aid in its proof). For this we need two classical results

- (i) *A lacunary trigonometric sum converges almost everywhere if and only if it is in  $L^2$ . A function in  $L^1$  with a lacunary Fourier expansion is in  $L^2$ . See e.g. [Z68] §5.6. Here lacunary means in Hadamard sense, i.e.  $q(k+1)/q(k) > 1 + c$ .*
- (ii) *If a trigonometric series converges pointwise on a set  $E$ , then both its positive and negative parts converge almost everywhere on  $E$ . This result is due to Plessner [P25]. A careful treatment can be found in [B64], §VIII.23, volume 2, page 151.*

To see that theorem 2 implies theorem 2' note that a function  $f$  which proves that  $\Lambda$  is not a Menshov spectrum also cannot be in  $\text{PLA} + \mathcal{L}_{-Q}$  as a decomposition  $f = g + h$ ,  $g \in \text{PLA}$ ,  $h \in \mathcal{L}_{-Q}$  carries over to a representation

$$f(t) = \sum_{n=0}^{\infty} \widehat{g}^{\text{PLA}}(n) e^{int} + \sum_{n=-\infty}^0 \widehat{h}(n) e^{int}$$

which converges almost everywhere since both its parts converge almost everywhere: the  $g$  part by definition of PLA and the  $h$  part because of (i).

Vice versa, assume by contradiction that  $\Lambda$  is a Menshov spectrum. Then every  $f$  has a representation as a sum  $\sum_{n \in \Lambda} c(n) e^{int}$  converging almost everywhere. But by Plessner's theorem the positive part converges a.e., so its limit, call it  $g$ , is a PLA function. Also the negative part converges a.e. so call its limit  $h$ . By the other direction of (i),  $h \in L^2$  and hence in  $\mathcal{L}_{-Q}$ . We get  $f = g + h$  with  $g \in \text{PLA}$  and  $h \in \mathcal{L}_{-Q}$  so, since  $f$  was arbitrary,  $\text{PLA} + \mathcal{L}_{-Q} = L^0$ . This shows that theorem 2' implies theorem 2, so they are equivalent.  $\square$

Satisfied that theorem 2 and 2' are equivalent, we start their proof. The first step is the following simple lemma.

**Lemma 3.** *Let  $Q$  satisfy (7). Then there is a number  $\alpha$ ,  $\frac{1}{3} < \alpha < \frac{2}{3}$  such that*

$$\{\alpha q(k)\} = o(1) \quad (18)$$

where  $\{x\}$  denotes the fractional part of  $x$ .

We remark that in §5 we will need that the estimate of  $\{\alpha q\}$  can be done uniformly in the superexponential growth of the  $Q$ , namely,

$$\{\alpha q(k)\} \leq C \max_{l \geq k} \left\{ \frac{q(l)}{q(l+1)} \right\}. \quad (19)$$

Also the restriction  $\alpha \in (\frac{1}{3}, \frac{2}{3})$  is only used in §5, here  $\alpha$  can be taken anywhere in  $(0, 1)$ .

*Proof.* We may assume without loss of generality that  $q(k+1)/q(k) > 2$  for all  $k$  (for  $k = 1$  we assume  $q(1) > 2$ ). Set

$$\alpha = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{\gamma(k)}{q(k)} \quad (20)$$

where the numbers  $0 < \gamma(k) \leq 1$  are to be defined. Assuming they are already defined for  $k \leq n$ , we denote by  $a(n)$  the  $n^{\text{th}}$  partial sums of the series (20) and set  $\gamma(n+1) := 1 - \{a(n)q(n+1)\}$  which implies that  $q(n+1)a(n+1)$  is integer. Continuing this process we get  $\alpha$ .

Now, for every  $n > 1$ :

$$\alpha q(n) = a(n)q(n) + q(n) \sum_{k>n} \frac{\gamma(k)}{q(k)}.$$

As already explained,  $a(n)q(n)$  is an integer. The second term is  $\leq q(n) \sum_{k>n} 1/q(k)$ , which is  $o(1)$  due to (7). This gives (18) and also the uniform estimate (19) remarked upon after the lemma.  $\square$

**Step 1.** With the lemma proved we can start the proof of theorem 2. Fix numbers  $d(n) > 0$  decreasing so fast that

$$\sum_{n>N} d^2(n) < \frac{d(N)^2}{N^2} \varepsilon^2 \left( \frac{N}{d(N)} \right), \quad (21)$$

where the function  $\varepsilon(n)$  was defined in lemma 2. Since this expression will repeat a lot, we will denote it for short by  $\varepsilon_N$ ,

$$\varepsilon_N := \frac{d(N)}{N} \varepsilon \left( \frac{N}{d(N)} \right), \quad (22)$$

so  $\sum_{n>N} d^2(n) < \varepsilon_N^2$ .

**Step 2.** Next use lemma 3 to find a number  $\beta \in (0, 2\pi)$  such that

$$e^{i\beta q(k)} \rightarrow 1. \quad (23)$$

**Step 3.** With  $\beta$  defined, one can find  $\nu(N)$  such that the following two properties hold,

$$|1 - e^{i\beta q(k)}| < \varepsilon_N \quad \forall k \text{ such that } q(k) > \nu(N) \quad (24)$$

$$|1 - e^{i\beta \nu(N)}| > 1. \quad (25)$$

These properties can be satisfied simultaneously because (24) is satisfied whenever  $\nu(N)$  is sufficiently large, while (25) is satisfied on a sequence converging to  $\infty$ . We now define

$$f(t) = \sum_{N=1}^{\infty} d(N) e^{-i\nu(N)t}. \quad (26)$$

This is the required function. As an aside we remark that it is in the Wiener algebra, but it might be highly non-smooth as we have no control over the relation between  $d(N)$  and  $\nu(N)$ .

**Step 4.** Recall now the discussion in the beginning of this section. We claim that  $f$  is a function demonstrating that  $\Lambda = \{-Q\} \cup \mathbb{Z}^+$  is not a Menshov spectrum, i.e. that  $f$  has no expansion

$$f(t) = \sum_n c(n) e^{int} \quad n \notin \Lambda \implies c(n) = 0 \quad (27)$$

which converges almost everywhere. Assume therefore by contradiction that an expansion (27) exists. Due to Plessner's theorem we know that  $\sum_{n<0} c(n) e^{int}$  converges (to some value), and since the negative part is lacunary we must have  $\sum_{n<0} |c(n)|^2 < \infty$ .

Somewhat similarly to the proof of theorem 1, we will now subtract the Fourier expansion of  $f$  and the non-standard one (27) and get a null series i.e. a trigonometric series converging to zero almost everywhere. Namely, define

$$\gamma(n) = c(n) - \hat{f}(n) = c(n) - \begin{cases} d(N) & n = -\nu(N) \\ 0 & \text{otherwise} \end{cases}$$

and get that

$$\sum_{n=-\infty}^{\infty} \gamma(n) e^{int} = 0 \quad \text{for almost every } t.$$

The crucial step is to examine  $f(t + \beta) - f(t)$  and the corresponding null series

$$\sum_{n=-\infty}^{\infty} \gamma(n) (e^{i\beta n} - 1) e^{int} = 0 \quad \text{for almost every } t. \quad (28)$$

As in the remark after (27), the positive and negative parts of (28) converge almost everywhere (not necessarily to zero).

**Step 5.** We will need some estimates for the  $L^2$  norm of the “tails” of (28) so let us state them now: for every  $N$ ,

$$\sum_{n < -\nu(N)} |\gamma(n) (e^{i\beta n} - 1)|^2 \leq C \varepsilon_N^2. \quad (29)$$

Here and below  $C$  may depend on  $\sum_{n < 0} |c(n)|^2$  (but not on  $N$ ).

*Proof of (29).*  $\gamma(n)$  can be non-zero only if  $n = -q(k)$  or if  $n = -\nu(k)$ . In the first case we have

$$|e^{i\beta n} - 1| = |e^{-i\beta q(k)} - 1| = |e^{i\beta q(k)} - 1| \stackrel{(24)}{<} \varepsilon_N$$

(recall that we are looking at  $n < -\nu(N)$  so (24) applies). All in all this gives

$$\sum_{k: q(k) > \nu(N)} |c(-q(k)) (e^{i\beta q(k)} - 1)|^2 < \varepsilon_N^2 \sum_{n < 0} |c(n)|^2$$

which we agreed to denote by  $C \varepsilon_N^2$ . The second kind of non-zero  $n$  is  $-\nu(k)$  and for this we simply use the definition of the  $d(k)$ , (21) and of  $f$ , (26), and get

$$\sum_{k > N} d(k)^2 |e^{i\beta \nu(k)} - 1|^2 \leq 4 \sum_{k > N} d(k)^2 \stackrel{(21)}{\leq} 4 \varepsilon_N^2.$$

Taking these two estimates together gives

$$\begin{aligned} \sum_{n < -\nu(N)} |\gamma(n) (e^{i\beta n} - 1)|^2 &= \sum_{n < -\nu(N)} |(c(n) + \widehat{f}(n)) (e^{i\beta n} - 1)|^2 \leq \\ &\leq \sum_{n < -\nu(N)} (2|c(n)|^2 + 2|\widehat{f}(n)|^2) \cdot |e^{i\beta n} - 1|^2 \end{aligned}$$

$$\text{By the above} \quad \leq \varepsilon_N^2 \cdot (2C + 8)$$

as needed. □

**Step 6.** We now proceed as in the proof of theorem 1 i.e. we wish to apply lemma 2 for some PLA function related to the null-series (28). We shift the null-series (28) by  $\nu(N)$  and divide it by  $d(N)(e^{-i\nu(N)\beta} - 1)$ . We get

$$\begin{aligned} q(t) &:= \frac{1}{d(N)(e^{-i\nu(N)\beta} - 1)} \sum_{n > -\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t} \\ &= 1 + \frac{1}{d(N)(e^{-i\nu(N)\beta} - 1)} \sum_{n < -\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t}. \end{aligned} \quad (30)$$

In other words, the first line is the “PLA expansion” of  $q$  and the second is the Fourier expansion. In particular  $q$  is a PLA function with  $\hat{q}^{\text{PLA}}(0) = 0$ . By (29),

$$\|1 - q\|_2 \leq \frac{C\varepsilon_N}{d(N)|e^{-i\nu(N)\beta} - 1|}.$$

By requirement (25),  $|e^{-i\nu(N)\beta} - 1| > 1$ , and with the definition of  $\varepsilon_N$  we get

$$\|1 - q\|_2 \leq \frac{C}{N} \varepsilon \left( \frac{N}{d(N)} \right).$$

Hence for  $N > C$  we may apply lemma 2 (to  $-q$ , but  $q^* = (-q)^*$ ) and get

$$\left| \left\{ t : q^*(t) > \frac{N}{d(N)} \right\} \right| > c_1.$$

Recalling that the PLA expansion of  $q$  is (30) we get a set of measure  $> c_1$  where

$$\sup_k \left| \sum_{n=-\nu(N)}^k \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t} \right| > d(N)|e^{-i\nu(N)\beta} - 1| \frac{N}{d(N)} > N. \quad (31)$$

**Step 7.** We only need to change the lower bound in the sum. But clearly

$$\sum_{n=-\nu(N)}^0 |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C$$

(if you want you can deduce this from (29) with the  $N$  there being 0). Using Markov’s inequality gives that the corresponding function cannot be large on a set of large measure:

$$\left| \left\{ t : \left| \sum_{n=-\nu(N)}^0 \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > C \right\} \right| < \frac{1}{2}c_1.$$

We subtract this from (31) and get a set of measure  $> \frac{1}{2}c_1$  where

$$\sup_k \left| \sum_{n=0}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > N - C.$$

Since  $N$  was arbitrary, we get a set of measure  $> \frac{1}{2}c_1$  where

$$\sup_k \left| \sum_{n=0}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| = \infty.$$

But this is exactly the positive part of (28). This is a contradiction since it was supposed to converge almost everywhere. This finishes the proof of theorem 2.  $\square$

**Conjecture.** *Probably theorem 2, and perhaps even theorem 3, hold for  $Q$  lacunary in Hadamard sense.*

There is another version of this problem. Let us introduce the concept of “Privalov spectrum”. We say that  $\Lambda$  is a Privalov spectrum if

$$\sum_{n \in \Lambda} c(n)e^{int} = 0 \quad \forall t \in E, \quad |E| > 0 \implies c(n) \equiv 0$$

Clearly, a Menshov spectrum can never be a Privalov spectrum. The trick of shifting by  $\beta$  employed above is useful also for this problem. For example,  $\Lambda = \{-2^n\}_{n=1}^\infty \cup \mathbb{Z}^+$  is a Privalov spectrum. To see this, it is enough to shift by  $\beta = 2^{-k}$  with  $k$  sufficiently large so as to satisfy  $E \cap (E + \beta) \neq \emptyset$ , and this reduces the result to the original Privalov theorem.

Thus a natural variation on the conjecture is: how sparse must  $Q$  be in order to ensure that  $-Q \cup \mathbb{Z}^+$  is a Privalov spectrum? This problem was considered by F. Nazarov in the early 90s in an unpublished work (private communication). The trick of shifting can be used to show that if  $Q$  is very fast increasing, then  $-Q \cup \mathbb{Z}^+$  is Privalov, but it seems that not under the condition (7) of superexponential growth. Faster growth of  $Q$  is necessary.

Another interesting generalization is to ask whether removing a superexponential sequence from a Menshov spectrum leaves one with a Menshov spectrum. Let us remark that a theorem of Talalyan [T69] shows that removing a single element from a Menshov spectrum will always result in a new Menshov spectrum.

## 5. PROOF OF THEOREM 3

The  $f$  demonstrating theorem 3 cannot be exactly as in the proof of theorem 2, as that  $f$  was lacunary itself! Hence it is itself in some  $\mathcal{L}_{-Q}$ , without the need to add any PLA function. It turns out that one can construct an  $f$  demonstrating

theorem 3 and very close to lacunary. We will construct an  $f \notin \text{PLA} + \mathcal{L}_{-Q}$  for any  $Q$  which is a sum of extremely lacunary *couples* of consecutive harmonics. The role of  $\beta$  (the value you shift by in the proof) in the theorem also changes — it has to be chosen after  $f$  is already known, so  $f$  cannot depend on it.

**Step 1.** To start the proof of theorem 3, we fix numbers  $d(n) > 0$  decreasing very fast. The precise condition will not make much sense now, so please do not dwell on it: it will become clearer in later stages of the proof. Precisely we define

$$\varepsilon\varepsilon_N := \frac{d(N)^2}{N^2} \varepsilon \left( \frac{N}{d(N)} \right) \varepsilon \left( \frac{N^2}{d(N)^2} \varepsilon^{-1} \left( \frac{N}{d(N)} \right) \right), \quad (32)$$

and then require  $d(n)$  to satisfy

$$\sum_{n>N} d^2(n) < \varepsilon\varepsilon_N^2. \quad (33)$$

Comparing to (21) we see that instead of using the function  $\varepsilon(n)$  from lemma 2 once, as we did in (21), here we need to iterate it. This is the reason for the notation  $\varepsilon\varepsilon_N$ .

**Step 2.** The choice of  $\nu$  now cannot depend on  $\beta$  as it is not yet known — it will instead depend on  $\ell$ , the rate at which  $q(k+1)/q(k)$  goes to infinity. Precisely, for every  $N$  find a  $\nu(N)$  such that

$$\ell(\nu(N)) > \frac{1}{\varepsilon\varepsilon_N} \quad (34)$$

where  $\ell$  is from the statement of theorem 3. We assume at this point that  $\ell$  is increasing, which we may, without loss of generality.

**Step 3.** With these we may define our function  $f$ ,

$$f(t) := \sum_{n=1}^{\infty} d(n) \left[ e^{-i(\nu(n)-1)t} + e^{-i\nu(n)t} \right]. \quad (35)$$

**Step 4.** We now need to show that  $f \notin \text{PLA} + \mathcal{L}_{-Q}$ , for any  $Q$ . Assume to the contrary that  $f = g + h$  with  $g \in \text{PLA}$  and  $h \in \mathcal{L}_{-Q}$  for some  $Q$  with  $q(k+1)/q(k) > \ell(q(k))$ . As in the proof of theorem 2 we denote by  $c(n)$  the coefficients of this “non-standard expansion” of  $f$ , i.e.  $c(n) = \widehat{g}^{\text{PLA}}(n)$  for  $n \geq 0$  and  $c(n) = \widehat{h}(n)$  for  $n < 0$ . Again we get a null series by subtracting the Fourier expansion of  $f$  and the non-standard one. Namely, define

$$\gamma(n) = c(n) - \begin{cases} d(k) & n = -\nu(k) + 1 \text{ or } n = -\nu(k) \\ 0 & \text{otherwise} \end{cases}$$

and get that

$$\sum_{n=-\infty}^{\infty} \gamma(n) e^{int} = 0 \quad \text{for almost every } t.$$

Next apply lemma 3 (and the remark following it) to find a number  $\beta \in (\frac{2\pi}{3}, \frac{4\pi}{3})$  such that

$$|e^{i\beta q(k)} - 1| < \frac{C}{\ell(q(k))} \quad \forall k. \quad (36)$$

(the  $C$  has two sources: the first is (19) and the second is the inequality  $|e^{2\pi it} - 1| \leq C\{t\}$ ). As before, the crucial step is to examine  $f(t + \beta) - f(t)$  and the corresponding null series

$$\sum_{n=-\infty}^{\infty} \gamma(n) (e^{in\beta} - 1) e^{int} = 0 \quad \text{for almost every } t. \quad (37)$$

Again this series not only converges to 0 symmetrically, also its positive part  $\sum_{n=0}^{\infty}$  and its negative part converge individually, almost everywhere, for the same reasons as before.

**Step 5.** We will need estimates for the  $L^2$  norm of the tails of (37), analogous to those of (29). Precisely,

$$\sum_{n < -\nu(N)} |\gamma(n) (e^{i\beta n} - 1)|^2 \leq C \varepsilon \varepsilon_N^2. \quad (38)$$

The proof is practically the same as that of (29), but we include it for the convenience of the reader.

*Proof.*  $c(n)$  can be non-zero only if  $n = -q(k)$  or if  $n = -\nu(k) + 1$  or  $-\nu(k)$ . In the first case we have

$$|e^{i\beta n} - 1| = |e^{-i\beta q(k)} - 1| = |e^{i\beta q(k)} - 1| \stackrel{(36)}{<} \frac{C}{\ell(q(k))}.$$

Now, we are looking at  $n < -\nu(N)$  so by the definition of  $\nu(N)$ , (34),

$$\ell(q(k)) \geq \ell(\nu(N)) \stackrel{(34)}{<} \frac{1}{\varepsilon \varepsilon_N}.$$

All in all this gives

$$\sum_{l: q(l) > \nu(k)} |c(-q(l)) (e^{i\beta q(l)} - 1)|^2 < C \varepsilon \varepsilon_N^2.$$



The second kind of non-zero  $n$  is  $-\nu(k) + 1$  and  $-\nu(k)$  and for this we simply use the definition of the  $d(k)$ , (33) and of  $f$ , (35), and get

$$\sum_{k>N} d(k)^2 \left( |e^{i\beta\nu(k)} - 1|^2 + |e^{i\beta(\nu(k)+1)} - 1|^2 \right) \leq 8 \sum_{k>N} d(k)^2 \stackrel{(33)}{\leq} 8\varepsilon\varepsilon_N^2.$$

Taking these two estimates together gives

$$\begin{aligned} \sum_{n<-\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 &= \sum_{n<-\nu(N)} |(c(n) + \widehat{f}(n))(e^{i\beta n} - 1)|^2 \leq \\ &\leq \sum_{n<-\nu(N)} (2|c(n)|^2 + 2|\widehat{f}(n)|^2) \cdot |e^{i\beta n} - 1|^2 \end{aligned}$$

$$\text{By the above} \quad \leq \varepsilon\varepsilon_N^2 \cdot (2C + 16). \quad \square$$

**Step 6.** We now proceed as in the proof of theorem 1 i.e. we wish to apply lemma 2 for some PLA function related to the null-series (37). Fix some  $N$  large and examine  $e^{-i(\nu(N)-1)\beta} - 1$  and  $e^{-i\nu(N)\beta} - 1$ . Since  $\beta \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$ , it is not possible for both numbers to be small. We therefore examine two cases:

- (i)  $|e^{-i\nu(N)\beta} - 1| > (d(N)/N)\varepsilon(N/d(N))$ .
- (ii)  $|e^{-i\nu(N)\beta} - 1| \leq (d(N)/N)\varepsilon(N/d(N))$ . This implies that  $|e^{-i(\nu(N)-1)\beta} - 1| > c$ .

Let us start with the first case (the other is similar but slightly simpler). We shift the null-series (37) by  $\nu(N)$  and get a new null-series whose positive part is the PLA expansion of some PLA function, and whose negative part is its Fourier expansion. Namely, define

$$\begin{aligned} q(t) &:= \frac{1}{d(N)|e^{-i\nu(N)\beta} - 1|} \sum_{n>-\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t} \\ &= 1 + \frac{1}{d(N)|e^{-i\nu(N)\beta} - 1|} \sum_{n<-\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t}. \end{aligned} \quad (39)$$

(the term “1” in the second line requires that  $N$  be sufficiently large because it requires that  $\nu(N) \notin Q$ . But this follows from our assumption (i) since if  $\nu(N) = q(k)$  then  $|e^{i\beta q(k)} - 1| < C\varepsilon\varepsilon_N^2$  which contradicts (i) for  $N > C$ ).

Now, by (38),

$$\sum_{n<-\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C\varepsilon\varepsilon_N^2.$$

Hence

$$\|q - 1\|_2 \leq \frac{C\varepsilon\varepsilon_N}{d(N)|e^{-i\nu(N)\beta} - 1|}$$

and since we assumed  $|e^{-iv(N)\beta} - 1| > (d(N)/N)\varepsilon(N/d(N))$  we get,

$$\|q - 1\|_2 \leq \frac{CN\varepsilon\varepsilon_N}{d(N)^2\varepsilon(N/d(N))}.$$

Recalling the definition of  $\varepsilon\varepsilon_n$  (32),

$$\|q - 1\|_2 \leq \frac{C}{N}\varepsilon \left( \frac{N^2}{d(N)^2}\varepsilon^{-1} \left( \frac{N}{d(N)} \right) \right)$$

and if  $N$  is sufficiently large the fraction is  $< 1$  and we can apply lemma 2. We get

$$\left| \left\{ t : q^*(t) > \frac{N^2}{d(N)^2}\varepsilon^{-1} \left( \frac{N}{d(N)} \right) \right\} \right| > c_1.$$

Recalling that the PLA expansion of  $q$  is given by (39) we get that there is a set of measure  $> c_1$  where

$$\begin{aligned} \sup_k \left| \sum_{n=1-\nu(N)}^k \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t} \right| &> \\ &> d(N)|e^{-iv(N)\beta} - 1| \frac{N^2}{d(N)^2}\varepsilon^{-1} \left( \frac{N}{d(N)} \right) > N \quad (40) \end{aligned}$$

where the second inequality again uses our assumption (i). We can replace in (40) the  $e^{i(n+\nu(N))t}$  by simply  $e^{int}$  as it does not change the absolute value of the expression. Finally to change the limit of the sum to 0 we note that clearly

$$\sum_{n=1-\nu(N)}^0 |c(n)(e^{i\beta n} - 1)|^2 \leq C.$$

This we may subtract from estimate (40) and get that on a set of measure  $> \frac{1}{2}c_1$ ,

$$\sup_k \left| \sum_{n=0}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > N - C \quad (41)$$

and we are done with case (i).

**Step 7.** We are left with case (ii) which is very similar, except that instead of shifting by  $\nu(N)$  we shift by  $\nu(N) - 1$ . The main reason to read this step is to see why we needed to define  $\varepsilon\varepsilon_N$  by iterating  $\varepsilon$  twice. As in case (i) for  $N$  sufficiently large we would have  $\nu(N) - 1 \notin Q$  so  $\gamma(-\nu(N) + 1) = d(N)$ . This gives, instead

of (39),

$$\begin{aligned} q(t) &:= \frac{1}{d(N)|e^{-i(\nu(N)-1)\beta} - 1|} \sum_{n>1-\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N)-1)t} \\ &= 1 + \frac{1}{d(N)|e^{-i(\nu(N)-1)\beta} - 1|} \sum_{n<1-\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N)-1)t}. \end{aligned}$$

The argument that  $\|q - 1\|_2$  is small is similar. We have

$$\sum_{n<1-\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C\varepsilon_N^2 + \frac{d(N)^2}{N^2}\varepsilon^2 \left( \frac{N}{d(N)} \right)$$

where the extra term is the one corresponding to  $n = -\nu(N)$  and is estimated by our assumption (ii). The extra term is the dominant one, so we may write

$$\sum_{n<1-\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C \frac{d(N)^2}{N^2} \varepsilon^2 \left( \frac{N}{d(N)} \right).$$

Since  $|e^{-i(\nu(N)-1)\beta} - 1| > c$  by our assumption, we get

$$\|q - 1\|_2 \leq \frac{1}{cd(N)} \cdot C \frac{d(N)}{N} \varepsilon \left( \frac{N}{d(N)} \right)$$

so again for  $N$  sufficiently large we may apply lemma 2 and get

$$\left| \left\{ t : q^*(t) > \frac{N}{d(N)} \right\} \right| > c_1$$

the same argument as in the previous case then shows that on a set of measure  $> c_1$ ,

$$\sup_k \left| \sum_{n=2-\nu(N)}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > cN$$

and again on a set of measure  $> \frac{1}{2}c_1$ ,

$$\sup_k \left| \sum_{n=0}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > cN - C. \quad (42)$$

As our conclusion (41) for case (i) is stronger, we in fact get that (42) holds regardless of whether case (i) or case (ii) held.

Since  $N$  was arbitrary, we see that on a set of measure  $> \frac{1}{2}c_1$  (the upper limit of the sets from (42)),

$$\sup_k \left| \sum_{n=0}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| = \infty.$$

In contradiction to our assumption after (37). Theorem 3 is thus proved.  $\square$

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